Band-limited Random Waveforms in Compressive Radar Imaging

Mahesh C. Shastry\textsuperscript{a}, Ram M. Narayanan\textsuperscript{a}, and Muralidhar Rangaswamy\textsuperscript{b}

\textsuperscript{a}The Pennsylvania State University, University Park, PA 16802; \textsuperscript{b}AFRL/RYRT Building 620, 2241 Avionics Circle, WPAFB, OH 45433

ABSTRACT

Compressive sensing makes it possible to recover sparse target scenes from under-sampled measurements when uncorrelated random-noise waveforms are used as probing signals. The mathematical theory behind this assertion is based on the fact that Toeplitz and circulant random matrices generated from independent identically distributed (i.i.d) Gaussian random sequences satisfy the restricted isometry property. In real systems, waveforms have smooth, non-ideal autocorrelation functions, thereby degrading the performance of compressive sensing algorithms. In this paper, we extend the existing theory to incorporate such non-idealities into the analysis of compressive recovery. The presence of extended scatterers also causes distortions due to the correlation between different cells of the target scene. Extended targets make the target scene more dense, causing random transmit waveforms to be sub-optimal for recovery. We propose to incorporate extended targets by considering them to be sparsely representable in redundant dictionaries. We demonstrate that a low complexity algorithm to optimize the transmit waveform leads to improved performance.

Keywords: Radar, compressive sensing, noise radar, circulant matrices, restricted isometry

1. INTRODUCTION

Noise radar\textsuperscript{1} refers to technology that employs continuous wave random noise waveforms for radar imaging. In recent years, ultrawide band (UWB) noise waveforms are being increasingly used.\textsuperscript{2} The UWB nature of the waveforms enables us to achieve high resolution in the context of conventional imaging. Noise radar is characterized by the simplicity of hardware\textsuperscript{2} compared to radar systems that use pulsed waveforms. The last few years have seen the development of digital noise radar systems.\textsuperscript{3} The digitization of signal processing has meant that systems are now portable and can be deployed for interesting imaging applications, such as through-the-wall imaging and human activity detection.

Compressive sensing refers to a new imaging paradigm which involves the recovery of signals that are sparse in some domain from a few non-adaptive measurements.\textsuperscript{4,5} In radar imaging, we desire to utilize waveforms with high bandwidths. Compressive sensing allows us to achieve higher resolutions than constrained by sampling hardware limitations. Radar in general, and noise radar in particular, have been shown to to benefit from advances in compressive sensing.\textsuperscript{6–9} The noise radar problem is interesting from a compressive sensing perspective because of the occurrence of partial circulant random matrices.\textsuperscript{7} The theoretical justification for compressive noise radar imaging is provided by the restricted isometry estimates for partial random circulant matrices.\textsuperscript{10,11}

The existing literature on this topic considers fairly idealized models of the compressive sensing problem. The transmitted waveform is modeled as a sequence of random statistically independent and identically distributed random variables and targets are modeled as consisting of point scatterers. However, in real systems, due to physical effects, the transmit and received waveforms are really bandlimited smoothed waveforms. The smoothing can be induced by the presence of low-pass filters at various stages in the hardware, and by the electromagnetic scattering effects of nonideal targets.\textsuperscript{12} In this paper, we propose specific models for this nonideality, analyze signal recovery performance, and propose approaches to mitigating the adverse effects.

Further author information: (Send correspondence to R.M.N.)
M.C.S, R.M.N.: E-mail: \{mcs312,rmn12\}@psu.edu, Telephone: +1 (814) 863 2602
M.R.: E-mail: muralidhar.rangaswamy@wpafb.af.mil

Compressive Sensing, edited by Fauzia Ahmad,
Proc. of SPIE Vol. 8365, 83650U © 2012 SPIE
CCC code: 0277-786X/12/$18 · doi: 10.1117/12.920936

Proc. of SPIE Vol. 8365 83650U-1

Downloaded From: http://proceedings.spiedigitallibrary.org/ on 07/14/2015 Terms of Use: http://spiedl.org/terms
2. COMPRESSIVE NOISE RADAR IMAGING

Compressive noise radar imaging involves inverting a standard linear equation. Given a target scene with characteristics defined by the transfer function \( s(t) \), the signal at the receiver \( y(t) \) can be modeled as the convolution of the transmitted signal \( x(t) \) with the target scene \( s(t) \). The received signal also consists of undesirable ambient noise, which we model as an additive white Gaussian random signal. Thus, we can write

\[
y(t) = \int_{-\infty}^{\infty} s(\tau)x(t-\tau)d\tau + \eta(t).
\]

Discretizing the above equation on a sufficiently fine grid, we get the vector equation

\[
y = Xs + \eta,
\]

with \( X \in \mathbb{R}^{N \times N} \) being a circulant matrix generated from the transmit-waveform. A circulant matrix is constructed by generating rows from circularly shifted copies of a single vector. Thus, the circulant matrix \( X \) generated from vector \( x = (x_0, x_1, ..., x_{N-1})^T \) will be

\[
X = \begin{pmatrix}
x_0 & x_{N-1} & \cdots & x_2 & x_1 \\
x_1 & x_0 & \cdots & x_3 & x_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{N-2} & x_{N-3} & \cdots & x_0 & x_{N-1} \\
x_{N-1} & x_{N-2} & \cdots & x_1 & x_0
\end{pmatrix}.
\]

The vectors are assumed to be real with \( y, s, \eta \in \mathbb{R}^N \). We define measurement as the act of projecting a given vector on to another set of vectors via dot products. Compressive sensing involves measuring the signal \( y \) with a measurement strategy denoted by the matrix \( \Phi \in \mathbb{R}^{M \times N} \). Thus, the measured signal \( z \in \mathbb{R}^M \), in compressive radar imaging, can be written as

\[
z = \Phi y = \Phi Xs + \Phi \eta.
\]

For compressive noise radar, we consider \( \Phi = R_{\Omega} \), where \( R_{\Omega} \) is a matrix composed of a sub-collection of the canonical basis functions with \( \Omega \subset [N] \). This measurement approach is known to work for the case where \( x(t) \) is a random waveform based on empirical\(^7\) and theoretical\(^10,11\) evidence. The proofs and evidence related to signal recovery implicitly assume that the elements of the vector \( x \) that generate \( X \) consists of elements which are i.i.d random. In the following section, we study the case where correlations are introduced between the elements of \( x \). We invert (4) for \( s \) by solving the optimization problem given by

\[
\begin{align*}
s^* &= \arg \min_{s \in \mathbb{R}^N} \|s\|_1 \\
\text{subject to} & \|z - \Phi Xs\|_2 \leq \sigma.
\end{align*}
\]

3. PARTIAL CORRELATED RANDOM CIRCULANT MATRICES

We consider the compressive noise radar imaging problem that occurs in two nonideal cases: (i) when the transmit waveform is bandlimited with by a known filter and (ii) when the target scene consists of extended scatterers. For mathematical purposes, the two situations can be treated as equivalent by including an extra term with the system matrix. We proceed by writing the transmitted waveform as

\[
\hat{x}(t) = x(t) \ast \psi(t),
\]

where \( x(t) \) is a fully random transmit waveform, \( \psi(t) \) represents the correlations, and the symbol \( \ast \) is defined as the convolution operator. For the discrete matrix operator associated with this function, we have

\[
\hat{X} = X\Psi.
\]
The matrix $\Psi$ is a circulant matrix generated from the coefficients of the bandlimiting filter $\psi(t)$. For deriving restricted isometry property (RIP) results, we incorporate this nonideality by modifying the system equation thus,

$$z = R_\Omega \tilde{X}s + R_\Omega \eta$$  \hspace{1cm} (9)

$$= R_\Omega X\Psi s + R_\Omega \eta.$$  \hspace{1cm} (10)

The recovery of the target scene and the solution to the convex optimization problem in (5) and (6) depends on the restricted isometry property of the matrix $R_\Omega X\Psi$.

### 3.1 Restricted Isometry Property (RIP)

The matrix $R_\Omega X\Psi$ is rank deficient. The existence of the solution to the compressive sensing optimization problem in (5) and (6) is decided by the restricted isometry property. A matrix $A$ is said to satisfy the restricted isometry property of order $S$ if for a small $\delta_S$,

$$(1 - \delta_S)\|p\|_{l_2}^2 \leq \|Ap\|_{l_2}^2 \leq (1 + \delta_S)\|p\|_{l_2}^2 \forall p \text{ with } \|p\|_{l_0} \leq S. \hspace{1cm} (11)$$

### 3.2 RIP for Circulant Matrices

In this paper, we adopt the approach of Rauhut et al.\textsuperscript{10} to understand the restricted isometry behavior of correlated circulant matrices. The restricted isometry property can be rewritten and simplified as

$$(1 - \delta_S)p^T p \leq p^T AA^T p \leq (1 + \delta_S)p^T p$$

$$-\delta_S \leq p^T (A^T A - I)p \leq \delta_S$$  \hspace{1cm} (13)

$$|p^T (A^T A - I)p| \leq \delta_S.$$  \hspace{1cm} (14)

Since $A$ is a fixed matrix, the above expression can be equivalently written as

$$\sup_{p \in \mathbb{R}^N, ||p||_{l_0} \leq S} |p^T (A^T A - I)p| = \delta_S. \hspace{1cm} (15)$$

Thus, we proceed by considering the matrix $A$. In our problem, $A = R_\Omega X\Psi$. We expand the matrix $X\Psi$ into a power series involving the permutation operator $S$, as follows

$$X = \sum_{k=1}^{N} x_k S^k, \hspace{1cm} (16)$$

$$H = \sum_{k=1}^{N} \psi_k S^k,$n

$$A = \left( \sum_{k=1}^{N} x_k S^k \right) \left( \sum_{k=1}^{N} \psi_k S^k \right), \hspace{1cm} (17)$$

$$A^T A = \sum_{k_1,k_2 \in [N]} \sum_{l_1,l_2 \in [N]} \psi_{k_1} \psi_{k_2} x_{l_1} x_{l_2} S^{-k_2-l_2-k_1-l_1} P_{10} S^{k_2+l_2+k_1+l_1}. \hspace{1cm} (18)$$

Alternately, we can also write $A^T A$ as

$$A^T A = \sum_{k,l \in [N] \times [N]} g_{k,l} S^{-k} P_{10} S^l,$$  \hspace{1cm} (19)

$$g_l \triangleq (x_l, \psi_l), \hspace{1cm} (20)$$

where, $x_l$ and $\psi_l$ refer to rows of the $X$ and $\Psi$ matrices. Following the analysis and notations of Rauhut et al.,\textsuperscript{10} the RIP analysis for correlated random matrices involves evaluating the mean and tail bounds on the process given by

$$\delta_S = \sup_{p \in \mathbb{R}^N, ||p||_{l_0} \leq S} |p^T (A^T A - I)p|.$$  \hspace{1cm} (21)
For convenience, we define the set, $T ≜ \{ p, \text{ such that } p \in \mathbb{R}^N \text{ with } ||p||_{l0} \leq S \}$. To study the restricted isometry property, we need to estimate the following two quantities:

$$\mathbb{E}\delta_S = \mathbb{E}\sup_{p \in T} |p^T(A^TA - I)p|,$$

$$\text{Prob}[\delta_S - \mathbb{E}\delta_S > \lambda] = \text{Prob}\left[\sup_{p \in T} |p^T(A^TA - I)p| - \mathbb{E}\delta_S > \lambda\right].$$

We proceed by defining $Z_p(k,l) = p^TS^{-1}k\Omega_s^l p$. We plug in the expressions from (20) into (22) and (23) to get

$$\mathbb{E}\delta_S = \mathbb{E}\sup_{p \in T} \left| \sum_{k,l} g_k g_l Z_p(k,l) \right|$$

$$> C_1 \mathbb{E}\sup_{p \in T} \left| \sum_{k,l} x_k x'_l Z_p(k,l) \right|,$$

$$\text{Prob}[\delta_S - \mathbb{E}\delta_S > \lambda] > C_2 \text{Prob}\left[\sup_{p \in T} \left| \sum_{k,l} x_k x'_l Z_p(k,l) \right| > \mathbb{E}\sup_{p \in T} \left| \sum_{k,l} x_k x'_l Z_p(k,l) \right| + \lambda\right],$$

where $x'_l$ are independent copies of $x_l$, and $C_1, C_2 > 1$ are constants. The above result follows by expanding the terms $\langle x_k, x'_l \rangle$ and then applying decoupling inequalities. The right hand sides of (26) and (27) are the estimates for RIP constants of partial uncorrelated random circulant matrices. The larger restricted isometry constants imply that the recovery results with correlated random circulant system matrices will be worse than the uncorrelated case, for a given level of sparsity and fixed number of measurements. When the correlations arise due to the presence of extended targets in the target scene, it makes sense to tweak the naively random transmit waveform to mitigate the effects of correlations, given prior knowledge of the nature of the target scene. In the following sections, we describe approaches to designing such optimal waveforms.

4. REDUNDANT DICTIONARIES IN IMAGING EXTENDED TARGETS

The RIP results demonstrate the suboptimality of the random waveform in the presence of correlations. In order to compensate for this nonideality, we optimize the waveform based on an $a$ priori model of non-ideality based on redundant dictionaries. Redundant dictionaries allow us to incorporate uncertain target information about the presence of extended targets. Extended targets reduce the sparsity of the target scene when represented in the canonical basis. However, using any basis other than the canonical basis for representing the target scene will result degrade the target image as suggested by (26) and (27). Thus, in order to represent jointly, both extended targets and point targets in an efficient manner, we propose to use a redundant dictionary that includes the identity matrix. If we follow the pulse-model given in (7), radar information about extended targets can be modeled as comprising of circulant matrix-molecules $\Psi^{(i)}$. The columns of $\Psi^{(i)}$ are often referred to as atoms. Each column of the matrix $\Psi^{(i)}$ thus represents a basis function (atomic basis) of the dictionary. The dictionary can thus be written as

$$\Psi = [\Psi^{(1)} | \Psi^{(2)} | ... | \Psi^{(D)}],$$

where each of $\Psi^{(i)} \in \mathbb{R}^{N \times N}$ with $i = 1, ..., D$ represents a molecule. The molecules are circulant matrices which are generated from the prior information of target profiles. The uncertainty is accounted for by increasing $D$. High uncertainty would require us to search through a larger dictionary, thus increasing the number of molecules. The compressive radar imaging problem can now be written as

$$s^* = \arg\min_{s \in \mathbb{R}^{DN}} ||s||_{l1} \text{ s.t. } ||z - R_\Omega Xs||_{l2} \leq \sigma.$$
The matrix $\Psi$ is designed to encode available information about the target scene. For instance, suppose we expect the target scene to contain a target with a spatial profile given by the general model, $\sigma(r) = f(kr + a)$, with parameters $a$ and $k$ representing the location and extent of the target, respectively. The dictionary can be extended to include the parameters $k$ and $a$. If we discretize the parameter space defined by $(k, a)$ into a grid of size $K_1 \times K_2$, we then have a dictionary that contains, $(K_1 + K_2)N$ elements to choose from. Thus, the estimated vector $\hat{s}^*$ will belong to the set $\mathbb{R}^{(K_1+K_2)N}$.

## 5. WAVEFORM DESIGN

The problem of designing optimal circulant sensing matrices was considered in a recent paper by Xu, et al.\textsuperscript{16} The approach is to minimize the cost function that is constructed to represent the mutual coherence\textsuperscript{4}

$$X^* = \text{arg min}_{x_{\text{circulant}}} \| \Psi^* X \Psi - I \|_F. \quad (30)$$

Algebraic simplification of the cost function results in\textsuperscript{16}

$$\| \Psi^* X \Psi - I \|_F = p^* \text{abs}(F^* \Psi \Psi F)p - 2p^* \text{diag}(F^* \Psi \Psi F) - N. \quad (31)$$

Thus we first solve the problem,

$$\mathcal{C}(p) = p^* \text{abs}(F^* \Psi \Psi F) - 2p^* \text{diag}(F^* \Psi \Psi F), \quad (32)$$

$$\hat{p}^* = \text{arg min}_{p \in \mathbb{R}^N} \mathcal{C}(p) \quad (33)$$

subject to $p \geq 0. \quad (34)$

The structure of the molecules allows us to simplify the cost function, and consequently the computation involved in solving the optimization problem. The simplification arises from the assumption that the molecules $\Psi_i$ are circulant. We can write each molecule as, $\Psi^{(i)} = FA^{(i)}F^*$. Thus, we can write the coefficient matrix of the quadratic cost function as

$$B = F^* [\Psi^{(1)} | \Psi^{(2)} | \cdots | \Psi^{(D)}] [\Psi^{(1)*} | \Psi^{(2)*} | \cdots | \Psi^{(D)*}]^* F \quad (35)$$

$$= F^* \left( \sum_{i=1}^{D} \Psi^{(i)} \Psi^{(i)*} \right) F \quad (36)$$

$$= F^* \left( \sum_{i=1}^{D} A^{(i)} F A^{(i)*} F^* \right) F \quad (37)$$

$$= \sum_{i=1}^{D} F^* F A^{(i)*} A^{(i)} F^* F \quad (38)$$

$$= \sum_{i=1}^{D} A^{(i)*} A^{(i)}. \quad (39)$$

The minimizer for the cost function will satisfy

$$\nabla \mathcal{C}(\hat{p}) = 0 \quad (40)$$

$$\text{abs} \left( \sum_{k=1}^{D} \hat{A}^{(k)*} \hat{A}^{(k)} \right) \hat{p} = \text{diag} \left( \sum_{k=1}^{D} \hat{A}^{(k)*} \hat{A}^{(k)} \right) \quad (41)$$

$$\hat{p}_i = \frac{1}{\sum_{k=1}^{D} \Lambda^{(k)*} \Lambda^{(k)}}. \quad (42)$$

We reconstruct the waveform by first modulating $\hat{p}$ with random phase, so that with $r_i \approx N(0, 1),

$$x_i = \sqrt{\hat{p}_i} e^{2\pi r_i}. \quad (43)$$
6. SIMULATIONS

In order to validate the above theory, simulations were performed to solve the one-dimensional radar imaging problem using optimized waveforms. The optimized waveform consists of 1024 samples. The target scene is represented in a redundant dictionary designed to encode the presence of extended targets. We consider the reflected signal has an SNR of 40 dB. The dictionary consists of eleven molecules. Ten of the molecules represent rectangular pulses of different extents. The basis functions that comprise each molecule matrices are time-shifted and inverted copies of the rectangular template, thus encoding the unknown locations of the targets. The eleventh basis function is the identity matrix, which is used to represent point scatterers in the target scene. A few examples of the basis functions used are plotted in Figure 1. The target scene consisted of extended targets and point targets. Compressive sensing recovery was performed using the spectral projected gradient algorithm for $l_1$-norms to solve the optimization problem, $\min_{s \in \mathbb{R}^D} ||s||_1$ subject to $||z - R_\Omega X_\Psi s||_2 < \sigma$. Only 25% of the samples of the reflected waveform were used for recovery. The reflected waveform was sampled uniformly. Since the target is not sparse in the canonical basis, a naive compressive sensing approach that does not use optimized waveforms fails to recover the target scene. The recovery performance when we use a transmit waveform that is optimized for the redundant dictionary model is shown in Figure 2. It is seen that the naive Gaussian random waveforms are unable to simultaneously recover extended and point targets. The reconstructed target scene is shown in Figure 2 and the corresponding vectors $s^*$ are also plotted. It is seen that optimized waveforms significantly outperform the naive approach.
For a statistical validation of the performance gain, we define the _miss-rate_, as the fraction of elements of $s^*$ that are recovered within $\nu = 10\%$ of the actual value. We can write this as

$$e(k) = \begin{cases} 0 & |s^*(k) - s(k)| / |s(k)| \leq \nu \\ 1 & \text{otherwise} \end{cases}.$$  \hspace{1cm} (44)

The recovery is repeated for 20 realizations. Each realization of the experiment consists of a target scene that is randomly generated using the modeling dictionary. The error metric $M$ is computed for each trial. This is plotted in Figure 3. It is seen that optimizing the waveform consistently yields an improvement in the performance of compressive recovery.

7. CONCLUSIONS

In this paper, we extended the theory of compressive radar imaging to account for correlated waveforms arising from extended targets. We showed analytically that correlations degrade the performance of compressive radar imaging. We proposed a model based on redundant dictionaries for incorporating the effects of correlations in compressive radar imaging. We presented a waveform optimization approach to mitigate correlation-effects that are introduced by extended targets. The low complexity of the waveform design algorithm makes it attractive for practical systems. The simulations demonstrate the benefits of optimizing the waveform to compensate for non-idealities. We are currently working on deriving the mutual coherence estimates for optimized transmit waveforms. In the future, we propose to use the theory of decoupling of random variables to quantify the effect of correlations on restricted isometry constants.

REFERENCES


